Topology Tools
for
Explainable and Green Artificial Intelligence

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Sixth EACA International School on Computer Algebra and its Applications

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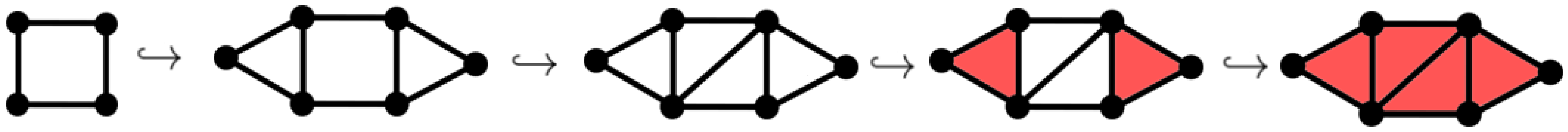
- Context: Green and Explainable artificial intelligence (REXASI-PRO)
- Computational topology tools: Persistent homology, barcodes, distance bottleneck, simplicial maps, Persistence modules, morphisms between persistence modules
- Partial matchings between barcodes
- Simplicial maps neural networks

Concepts from computational topology

- (abstract) Simplicial complexes
- Homology
- Vertex map, simplicial map
- Simplicial Approximation Theorem
- Category SpCpx : simplicial complexes, simplicial maps
- Filtration
- Persistent homology
- **Persistence module**



$$H_1(K_1) \longrightarrow H_1(K_2) \longrightarrow H_1(K_3) \longrightarrow H_1(K_4) \longrightarrow H_1(K_5)$$

$$V = PH_1(K)$$

$$0 \longrightarrow V_1 \xrightarrow{\rho_1^2} V_2 \xrightarrow{\rho_2^3} V_3 \xrightarrow{\rho_3^4} V_4 \xrightarrow{\rho_4^5} V_5 \longrightarrow 0$$

persistence module

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- **Persistence module**

A persistence module can be seen as a functor from {1, … , n} to the category of vector spaces

This definition can be extended to any other totally ordered set

$$(V_t, \rho_p^q)$$

$$1 \leq 2 \leq \dots \leq n$$

$$\Downarrow$$

$$0 \xrightarrow{\rho_0^1} V_1 \xrightarrow{\rho_1^2} V_2 \xrightarrow{\rho_2^3} \dots \xrightarrow{\rho_{n-1}^n} V_n \xrightarrow{\rho_n^{n+1}} 0$$

- $\rho_q^l \rho_p^q = \rho_p^l$ if $0 \leq p \leq q \leq l \leq n+1$
- ρ_p^p is the identity map

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- **Persistence module**

The category of persistence modules satisfies that

- The **direct sum** of persistence modules is a persistence module
- The **intersection** of persistence modules is a persistence module
- The **quotient** of persistence modules is a persistence module
- The notion of **submodule** is well defined

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- **Persistence module**

$$\boxed{V \simeq \bigoplus_{I \in S} \left(\oplus_{m_I} k_I \right)}$$

$B = \{(I, m_I)\}$

$S = \{ \mathrm{I} \}$

Interval module $k_{It} = \begin{cases} 1 & \text{if } t \in I \\ 0 & \text{otherwise} \end{cases}$

$I = [a, b]$ $\quad t \in I$

$\mathrm{Im}^+_{at}(V) := \mathrm{im}\ \rho^t_a,$ $\qquad \mathrm{Ker}^+_{bt}(V) := \ker \rho^{b+1}_t$

$\mathrm{Im}^-_{at}(V) := \mathrm{im}\ \rho^t_{a-1},$ $\qquad \mathrm{Ker}^-_{bt}(V) := \ker \rho^b_t$

$$\boxed{k_I \simeq \frac{V^+_{It}}{V^-_{It}}} \qquad \boxed{\mathrm{m_I} = \dim \frac{V^+_{It}}{V^-_{It}}}$$

$$V^+_{It} := \mathrm{Im}^+_{at}(V) \cap \mathrm{Ker}^+_{bt}(V)$$

$$V^-_{It} := \mathrm{Im}^-_{at}(V) \cap \mathrm{Ker}^+_{bt}(V) + \mathrm{Im}^+_{at}(V) \cap \mathrm{Ker}^-_{bt}(V)$$

Example: $B = \{([1,4], 1), ([2,3], 2), ([3,4], 1)\}$

$Rep\ B = \langle [1,4], [2,3], [2,3], [3,4] \rangle$

$\mathrm{SS} = \{[1,4], [2,3], [3,4]\}$

Concepts from computational topology

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- Persistence module
- **Morphism between persistence modules**

$$\begin{array}{ccc} U \\ f\big\uparrow \\ V \end{array} \qquad \begin{array}{ccccccc} U_1 & \longrightarrow & U_2 & \longrightarrow & \cdots & \longrightarrow & U_n \\ f_1\big\uparrow & & f_2\big\uparrow & & & & f_n\big\uparrow \\ V_1 & \longrightarrow & V_2 & \longrightarrow & \cdots & \longrightarrow & V_n \end{array}$$

Example:

- Let $\mathbb{X}$ and $\mathbb{Y}$ be two finite subsets from $\mathbb{R}^n$ such that $\mathbb{X} \subseteq \mathbb{Y}$.
- This induces an embedding $\mathrm{VR}(\mathbb{X}) \hookrightarrow \mathrm{VR}(\mathbb{Y})$.
- In turn, this induces a persistence morphism $f : V \to U$, where $V = \mathrm{PH}_n(\mathrm{VR}(\mathbb{X}))$ and $U = \mathrm{PH}_n(\mathrm{VR}(\mathbb{Y}))$

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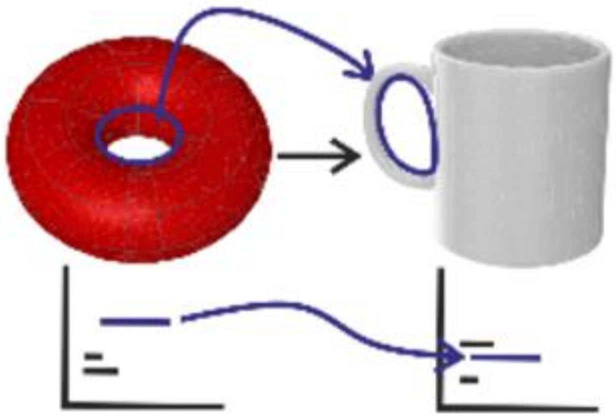

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- Partial matchings between barcodes
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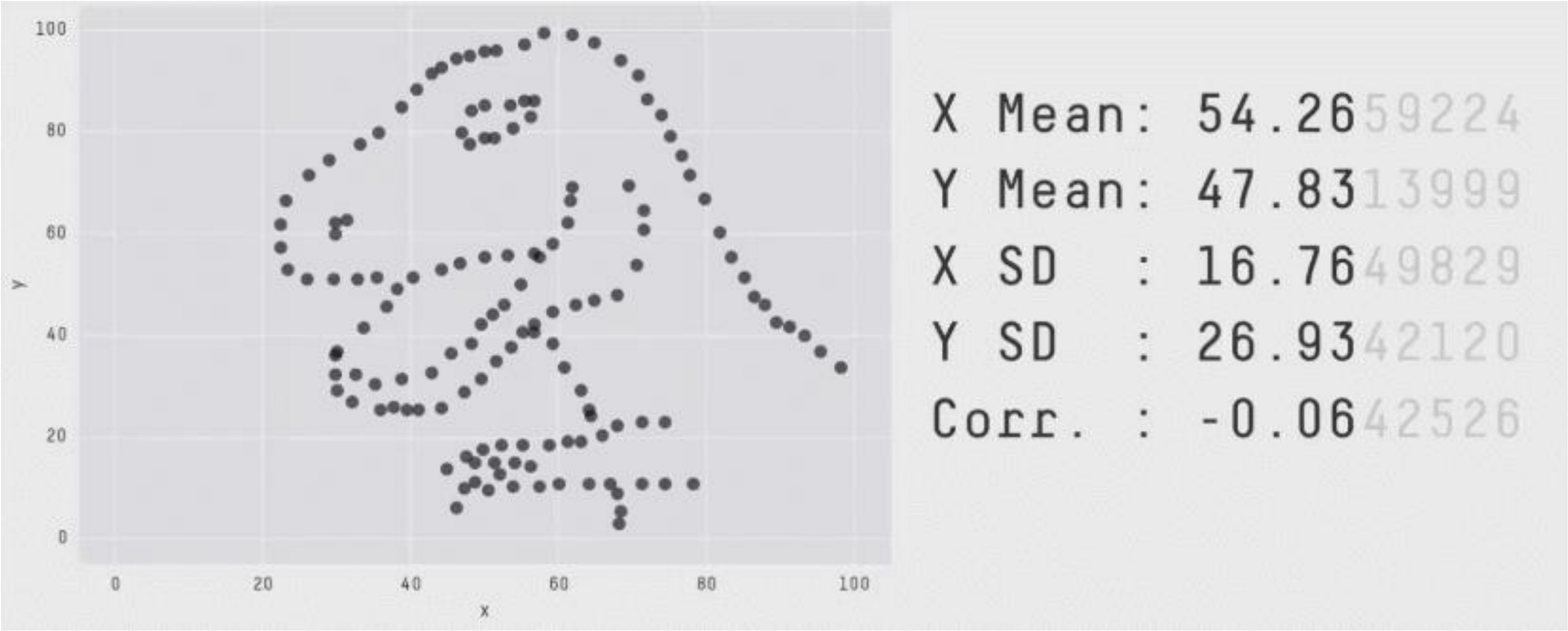
Partial matchings between barcodes

Joint work with Manuel Soriano-Trigueros and Álvaro Torras-Casas

Partial matchings between barcodes

A motivation

Stats is not enough:

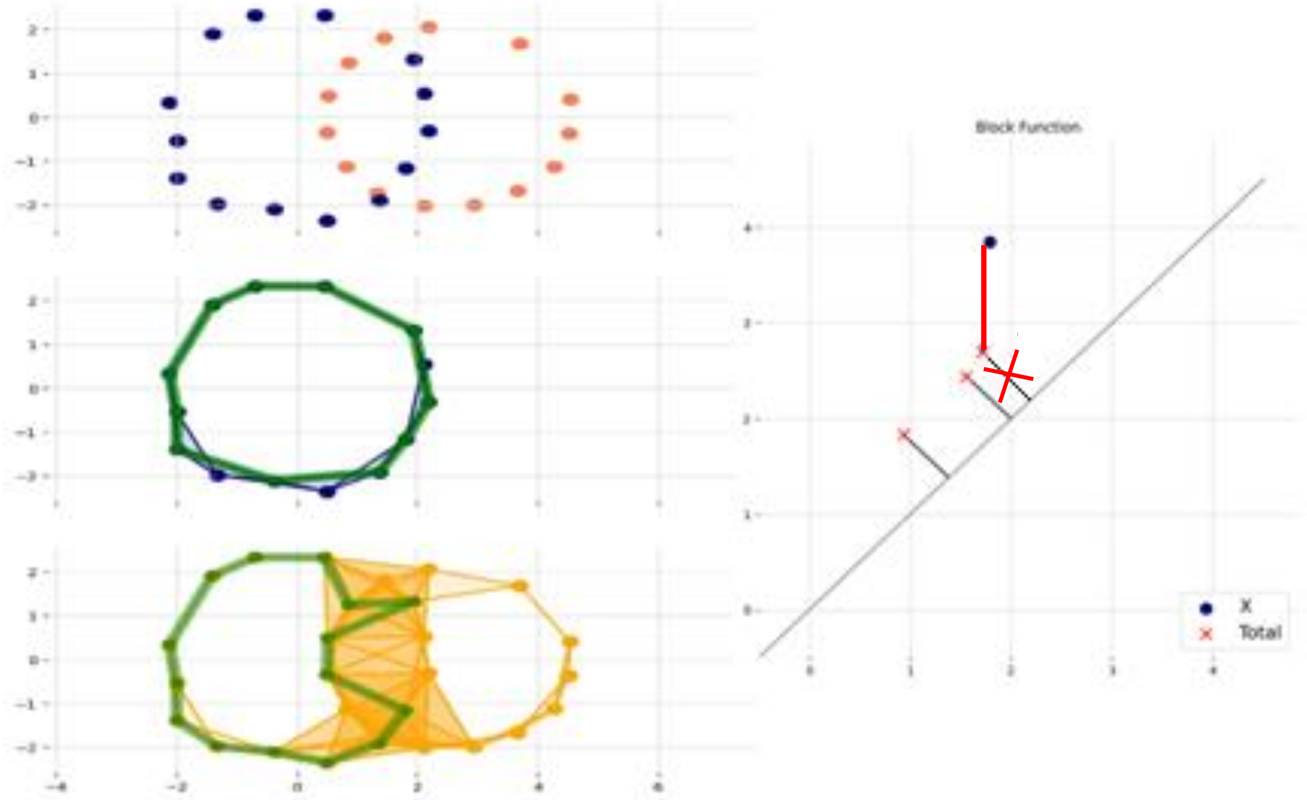


Datasaurus dozen dataset

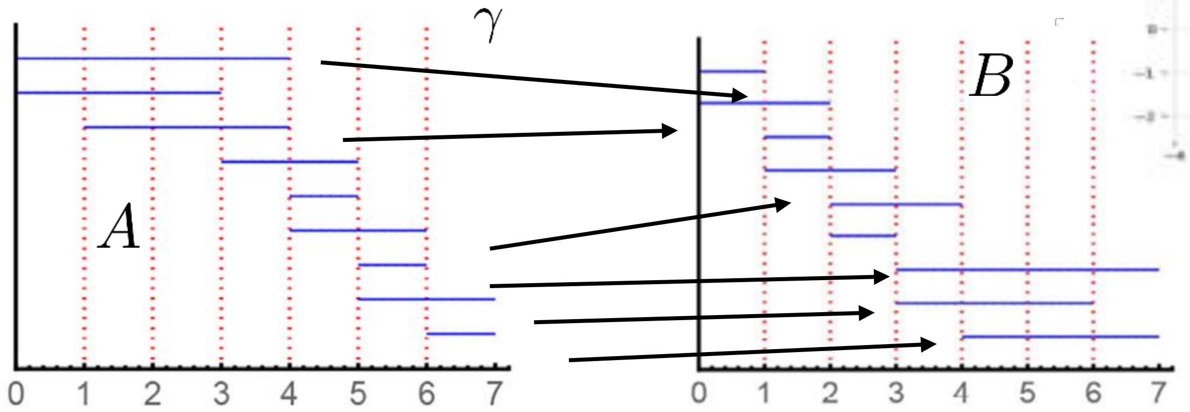
Partial matchings between barcodes

A motivation

Bottleneck distance is not enough:



RECALL:



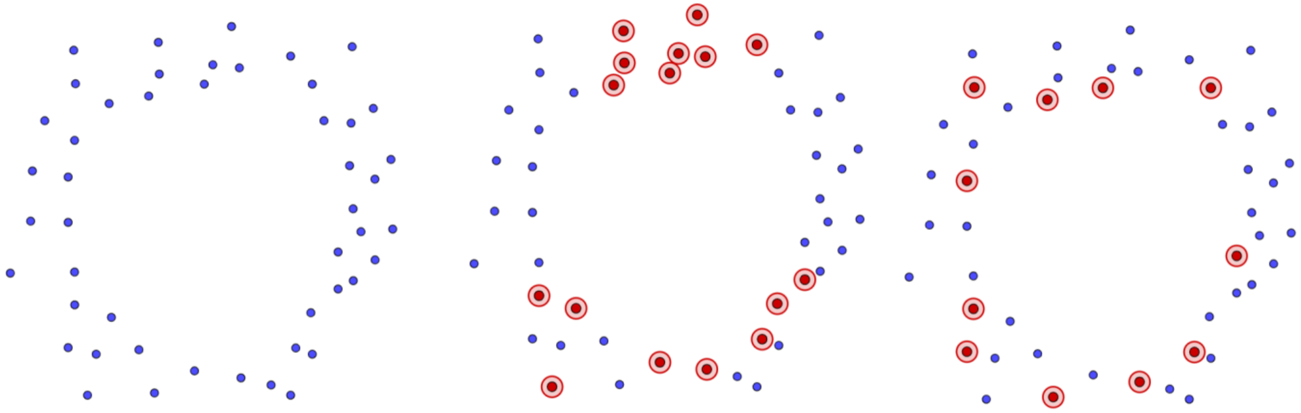
Bottleneck distance: $d_\infty(A,B) = \min_{\gamma} \; d_{\infty\gamma}(A,B)$

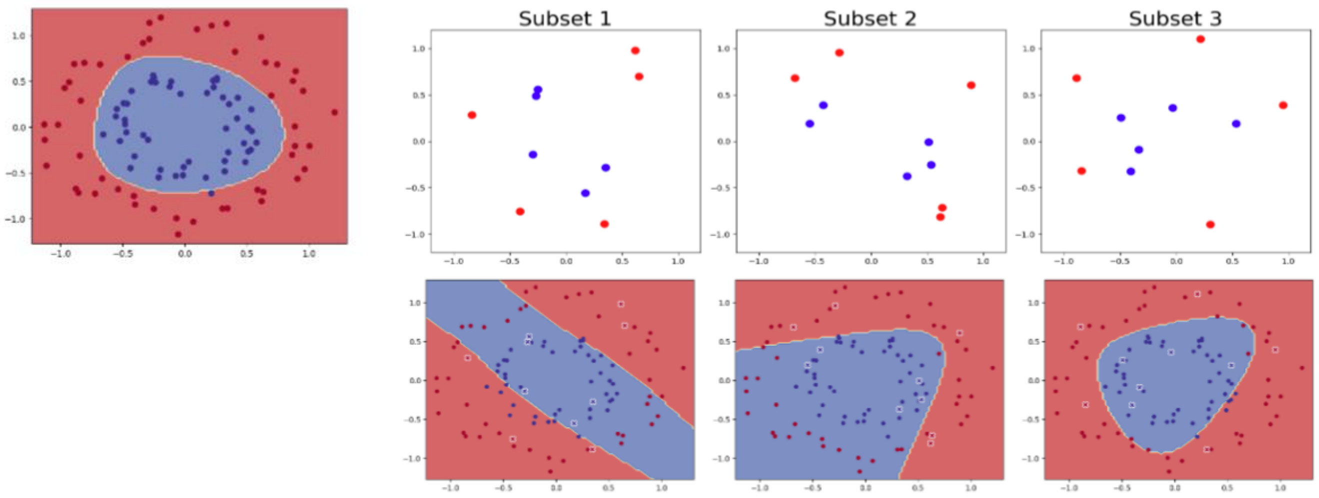
Partial matchings between barcodes

A motivation

When can we say that a subset Y of a given a dataset X "samples" the same continuous space than X?

Why is it important?
- Data hungry
- Energy saving
- Storage saving
- Data is expensive





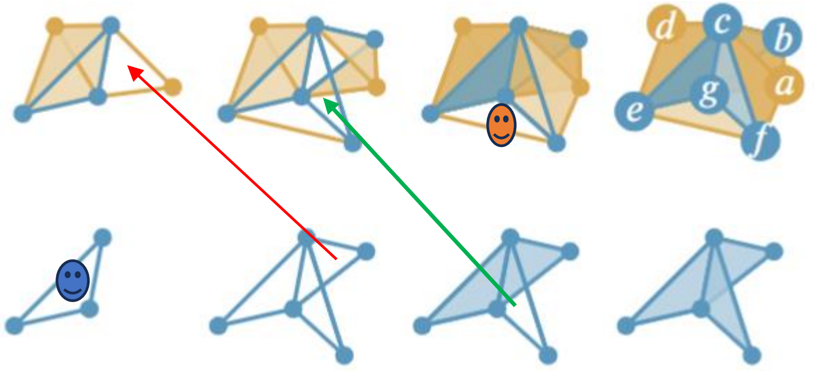
Partial matchings between barcodes

Example:

K

∪

L



$$
\begin{tikzcd}
PH_1(K) \arrow[d, phantom, "\parallel"] & & & & & & \\
U & 0 \arrow[r] & U_1 \arrow[r] & U_2 \arrow[r] & U_3 \arrow[r] & U_4 \arrow[r] & 0 \\
V \arrow[u, "f"] \arrow[d, phantom, "\parallel"] & 0 \arrow[r] & V_1 \arrow[r] \arrow[u] & V_2 \arrow[r] \arrow[u] & V_3 \arrow[r] \arrow[u] & V_4 \arrow[r] \arrow[u] & 0 \\
PH_1(L) & & & & & &
\end{tikzcd}
$$

$$PH_1(K) \simeq k_{[1,2]} \oplus k_{[2,2]} \oplus k_{[2,3]}$$

$$PH_1(L) \simeq k_{[1,2]} \oplus k_{[2,2]} \oplus k_{[2,3]}$$

$$\boxed{d_\infty\big(B(U), B(V)\big) = 0}$$

Partial matchings between barcodes

$\boxed{S_V = \{ I \}}$ $\boxed{S_U = \{ J \}}$

Given $B(V) = \{(I, m_I)\}$ and $B(U) = \{(J, m_J)\}$,
we define a ~~**partial matching**~~ satisfying
block function

$$\mathcal{M} : S_V \times S_U \longrightarrow \mathbb{Z}_{\geq 0}$$

$$\sum_{J \in S_V} \mathcal{M}(I, J) \leq m_I$$

The number of arrows leaving from an interval is smaller than its multiplicity

$$\cancel{\sum_{I \in S_U} \mathcal{M}(I, J) \leq n_J}$$

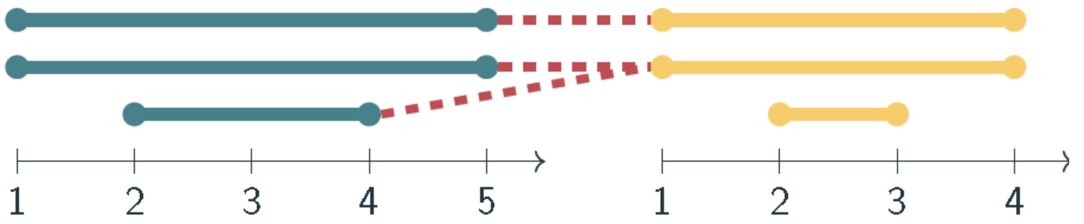
~~The number of arrows arriving to an interval is smaller than its multiplicity~~

Partial matchings between barcodes

Block function: $\mathcal{M}: S_V \times S_U \to \mathbb{Z}_{\geq 0}$ $\quad \sum_{J \in S_U} \mathcal{M}(I, J) \leq m_I$

Example:

B(V) $= \{([2,4],1),([1,5],2)\}$ and B(U) $= \{([2,3],1),([1,4],2)\}$



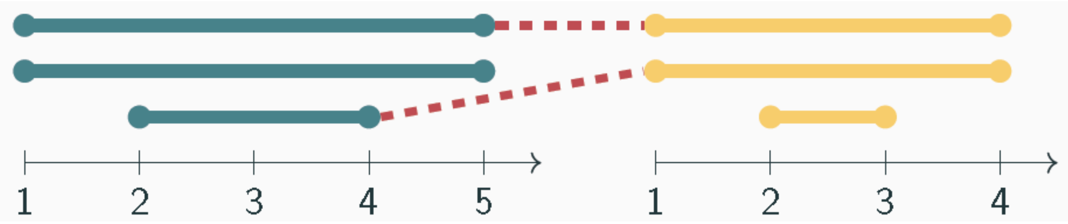
Consider $\mathcal{M}$ is zero except for

$$\mathcal{M}([2,4],[1,4]) = 1 \text{ and } \mathcal{M}([1,5],[1,4]) = 2\,.$$

We can always obtain a partial matching γ from a block function

$\mathcal{M}$ is a block function, since

$$\mathcal{M}([2,4],[1,4]) = 1 \leq m_{[2,4]} \text{ and } \mathcal{M}([1,5],[1,4]) = 2 \leq m_{[1,5]}$$



$\mathcal{M}$ is not a partial matching

$$\mathcal{M}([2,4],[1,4]) + \mathcal{M}([1,5],[1,4]) = 3 \not\leq n_{[1,4]} = 2\,.$$

$$\sum_{J \in S_2} \gamma(I, J) \leq m_I \qquad \sum_{I \in S_1} \gamma(I, J) \leq n_J$$

Partial matchings between barcodes

The induced block function: $\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0}$ $\quad \sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$

$$V \xrightarrow{\quad f \quad} U$$

$$(I, m_I) \xrightarrow[\mathcal{M}_f(I,J)\,?]{} (J, m_J)$$

Partial matchings between barcodes

The induced block function: $\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0}$ $\quad \sum_{J \in S_U} \mathcal{M}_f(I,J) \leq m_I$

Partial Matchings Induced by Morphisms between Persistence Modules. Computational Geometry, 112, 101985 (2023)

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[d] & & U \arrow[d] \\ (I, m_I) \arrow[rr, "\mathcal{M}_f(I,J)"'] & & (J, m_J) \end{tikzcd}$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \ X_{IJt}}$$

$t \in I \cap J$

$$\begin{tikzcd} V^+_{It}/V^-_{It} \arrow[dr] & & U^+_{Jt}/U^-_{Jt} \arrow[dl] \\ & X_{IJt} & \end{tikzcd}$$

$$X_{IJt} = \frac{fV^+_{It} \cap U^+_{Jt}}{fV^-_{It} \cap U^+_{Jt} + V^+_{It} \cap U^-_{Jt}}$$

Partial matchings between barcodes

The induced block function: $\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0}$ $\quad \sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$

Let $I = [a, b]$ and $J = [c, d]$.

$$\boxed{\begin{array}{ll} \mathcal{M}_f(I, J) = \dim X_{IJd} & \text{if } J \leq I \\ \mathcal{M}_f(I, J) = 0 & \text{otherwise} \end{array}}$$

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We say that $J \leq I$ if $c \leq a \leq d \leq b$.

$$X_{IJt} = \frac{fV_{It}^{+} \cap U_{Jt}^{+}}{fV_{It}^{-} \cap U_{Jt}^{+} + V_{It}^{+} \cap U_{Jt}^{-}}$$

Partial matchings between barcodes

The induced block function: $\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0}$ $\quad \sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$

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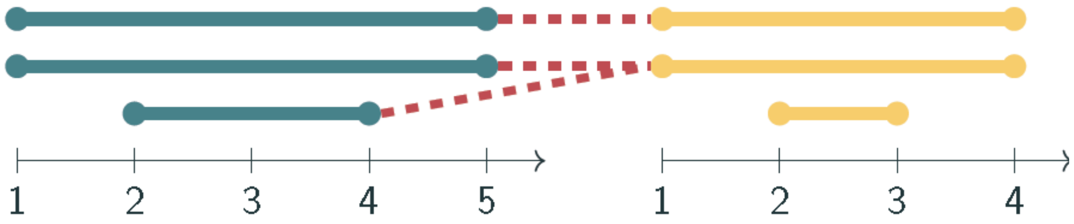
Nested intervals:



- $[a, b]$ and $[c, d]$ are nested if $a < c < d < b$

Prop.: If for any set of intervals $S \subseteq S_V$ we have that

$$\sum_{I \in S} \mathcal{M}_f(I, J) > n_J ,$$

then there exists a pair of nested intervals in S_V.

Corollary: If there are no two nested intervals in S_V then M_f is a partial matching.

Partial matchings between barcodes

Linearity of the induced block function:

Partial Matchings Induced by Morphisms between Persistence Modules. Computational Geometry, 112, 101985 (2023)

Given a direct sum of morphisms of persistence modules

$$f^1 \oplus f^2 : V^1 \oplus V^2 \to U^1 \oplus U^2$$

we have that

$$\mathcal{M}_{f^1 \oplus f^2}(I, J) = \mathcal{M}_{f^1}(I, J) + \mathcal{M}_{f^2}(I, J)$$

Example:

$$\begin{tikzcd} U & k \arrow[r, "\mathrm{Id}"] & k \arrow[r] & 0 \\ V \arrow[u, "f"] \arrow[u, phantom, shift right=20pt, "\simeq"] & 0 \arrow[u] \arrow[r] & k^2 \arrow[u, "\begin{pmatrix}0\\1\end{pmatrix}"'] \arrow[r, "\begin{pmatrix}1\\0\end{pmatrix}"'] & k \arrow[u] \end{tikzcd}$$

$$\mathcal{M}_f \ ?$$

$$\begin{tikzcd} U & 0 \arrow[r] & 0 \arrow[r] & 0 & k \arrow[r, "\mathrm{Id}"] & k \arrow[r] & 0 \\ V \arrow[u, "f"] \arrow[u, phantom, shift right=20pt, "\simeq"] & 0 \arrow[u] \arrow[r] & k \arrow[u] \arrow[r, "\mathrm{Id}"] & k \arrow[u] \arrow[r, phantom, "\oplus"] & 0 \arrow[u] \arrow[r] & k \arrow[u, "\mathrm{Id}"'] \arrow[r] & 0 \arrow[u] \end{tikzcd}$$

$$\{([1,2],1)\}$$

$$\uparrow$$

$$\{([2,3],1),([2,2],1)\}$$

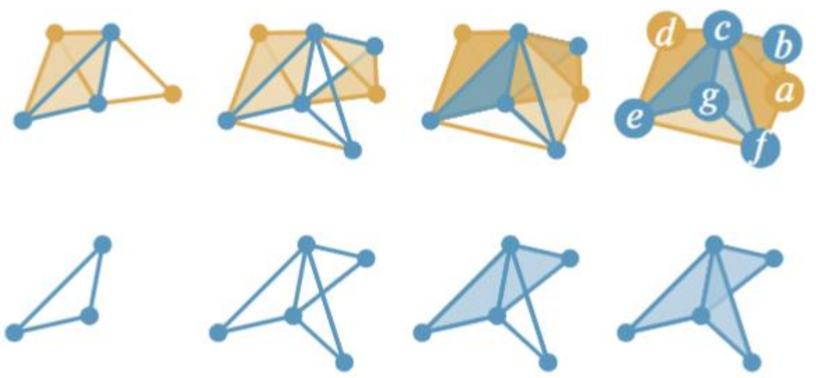
Partial matchings between barcodes

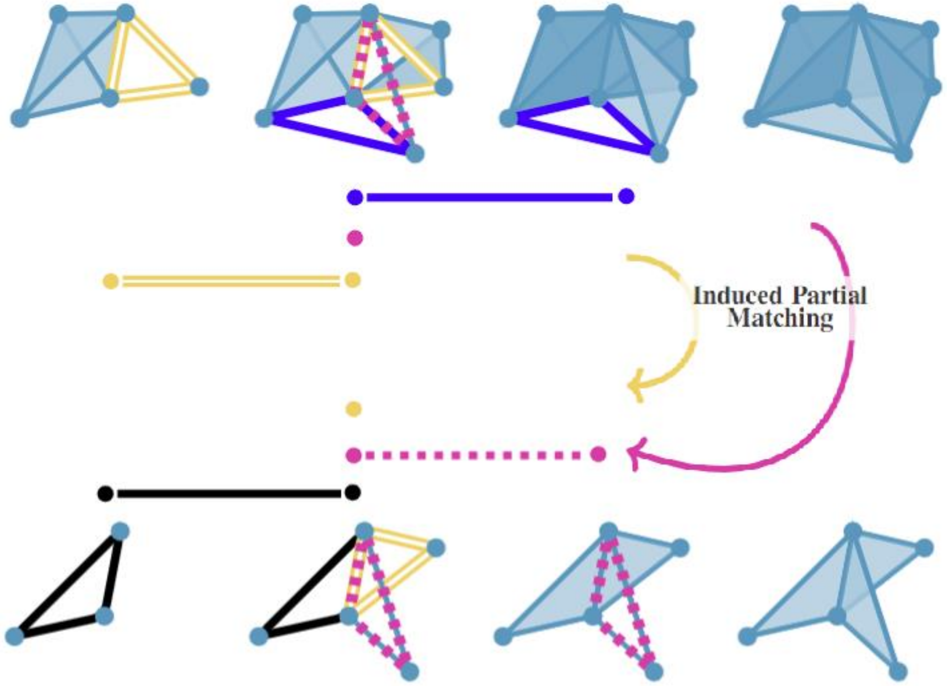
Example:

K

∪

L

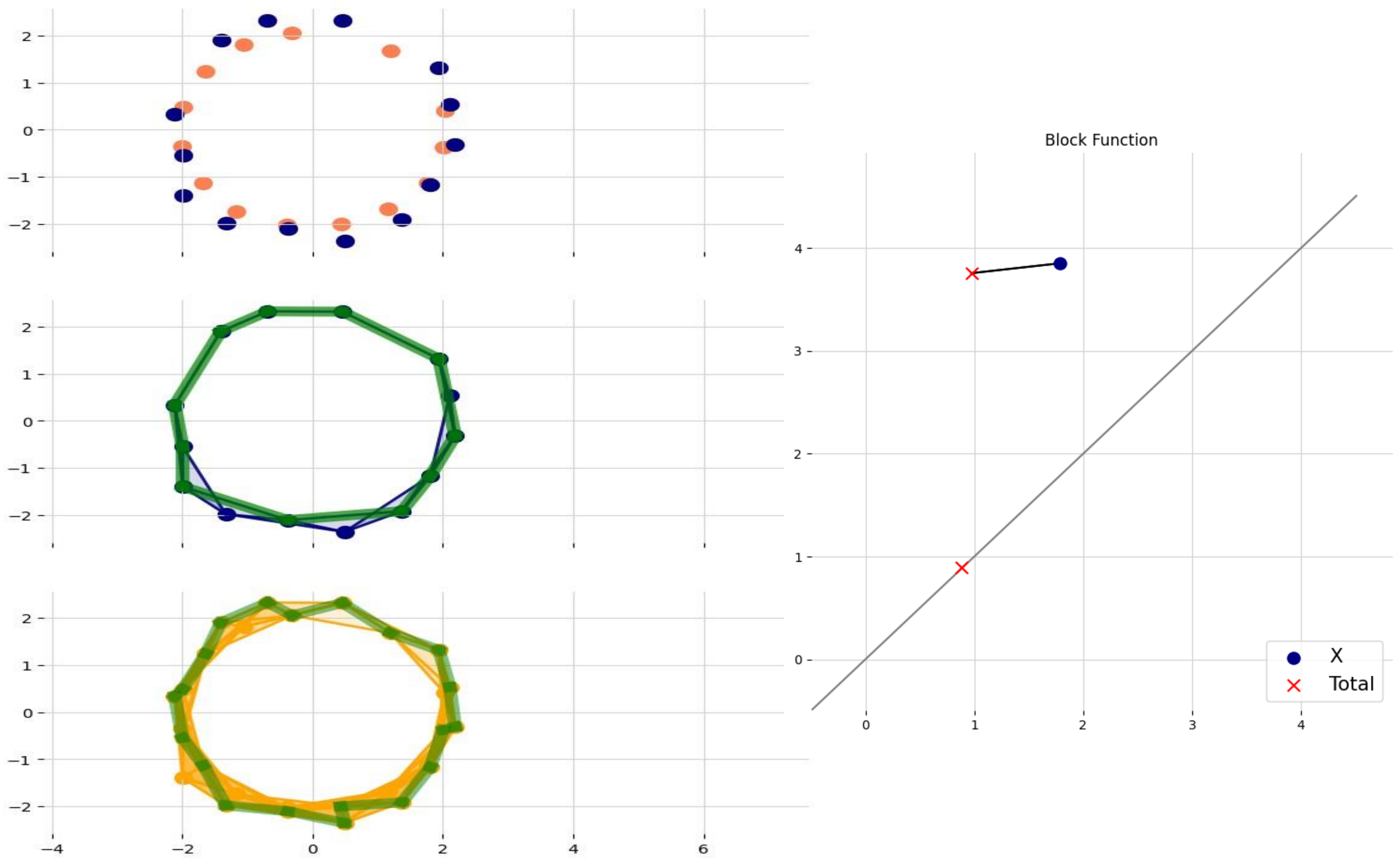




$$PH_1(K) \simeq k_{[1,2]} \oplus k_{[2,2]} \oplus k_{[2,3]}$$

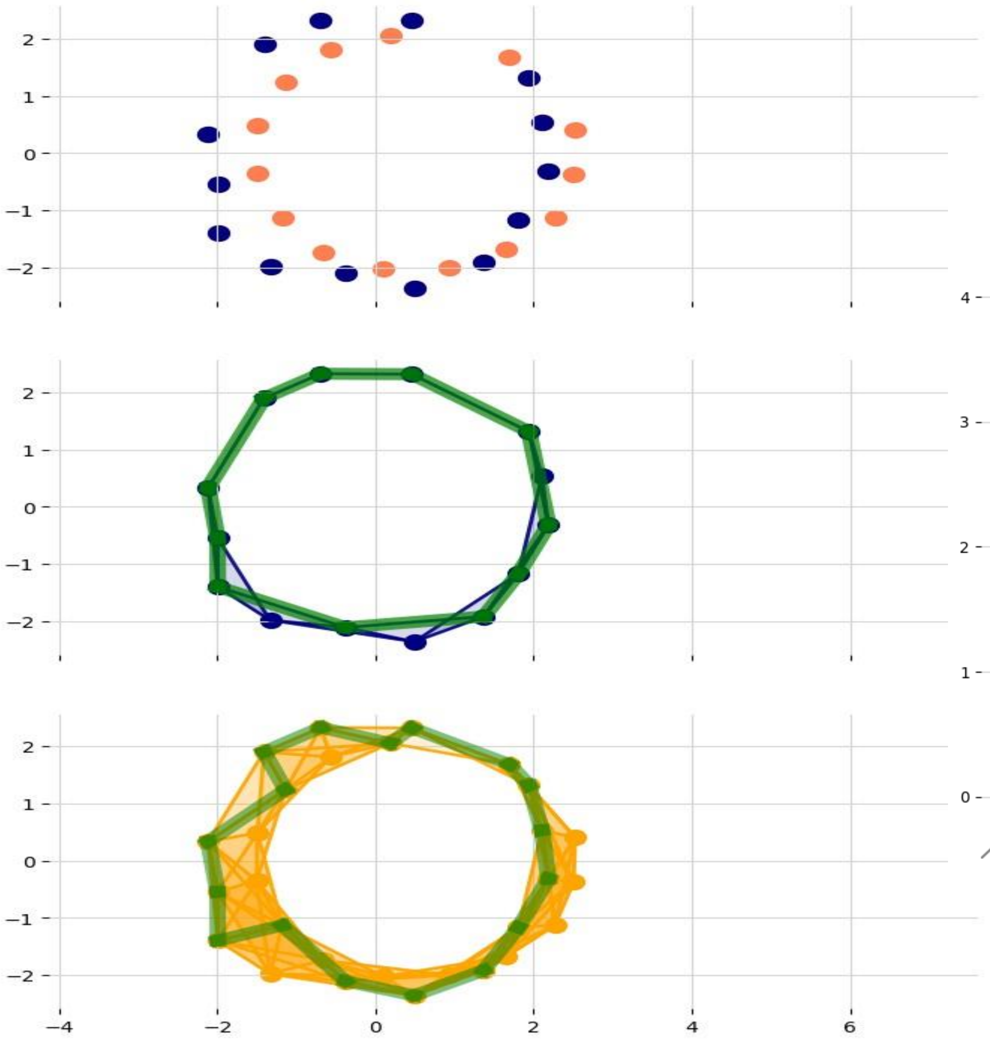
$$PH_1(L) \simeq k_{[1,2]} \oplus k_{[2,2]} \oplus k_{[2,3]}$$

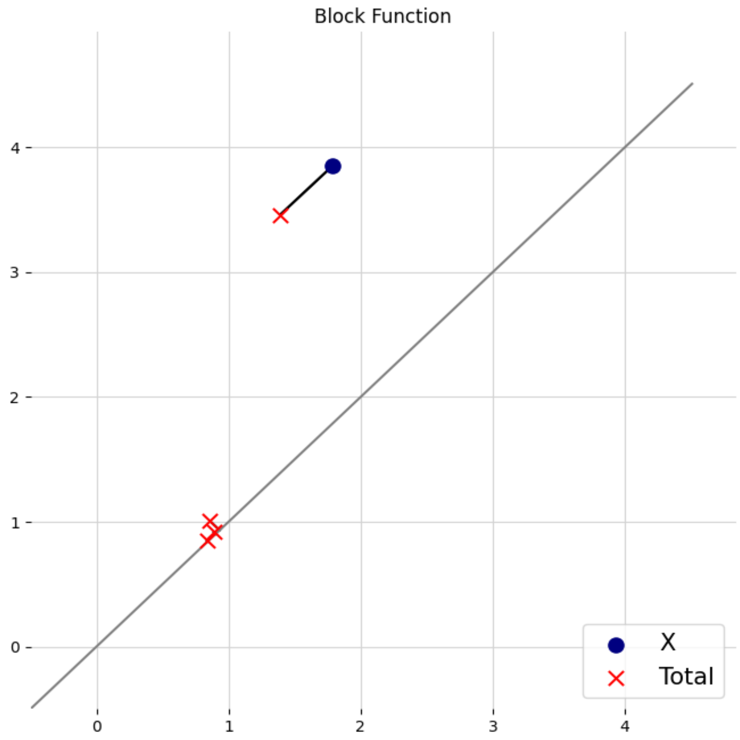
Partial matchings between barcodes



Matching circles in the plane

Partial matchings between barcodes





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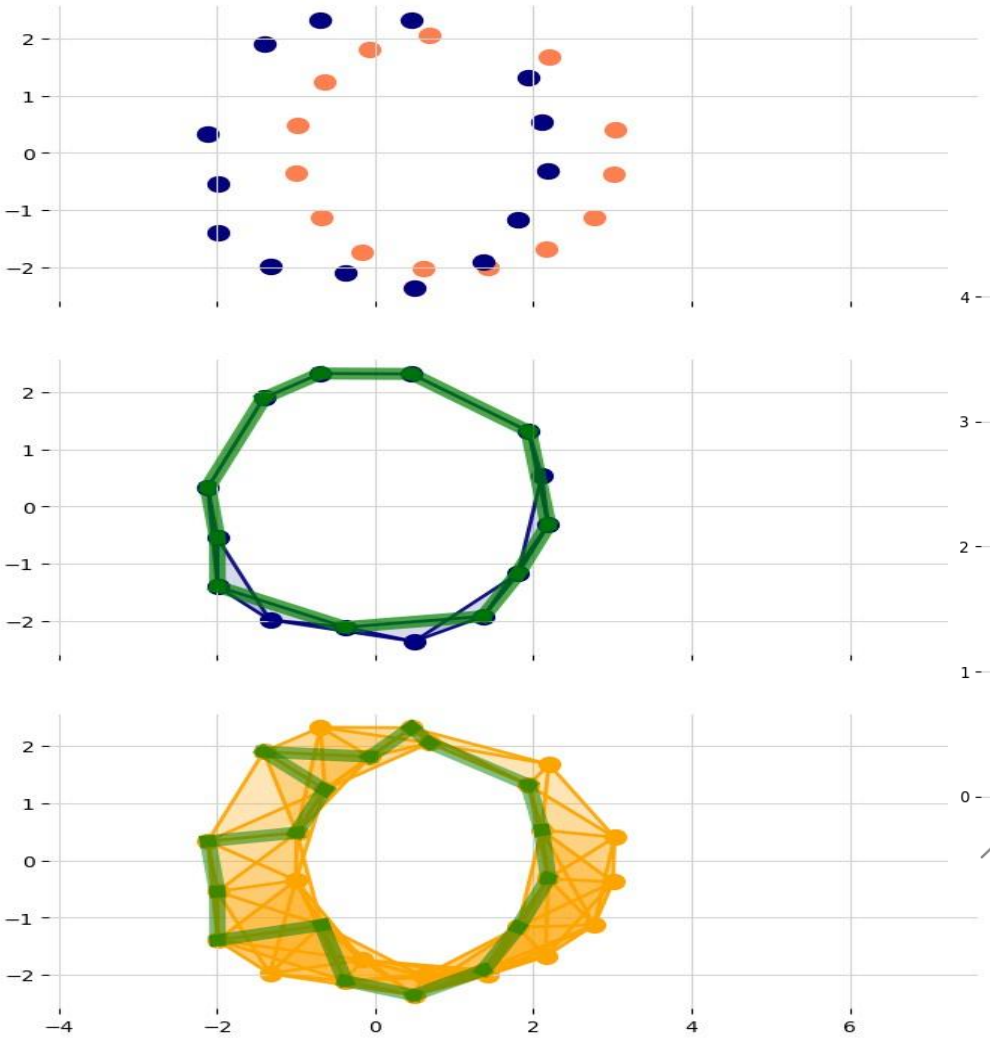

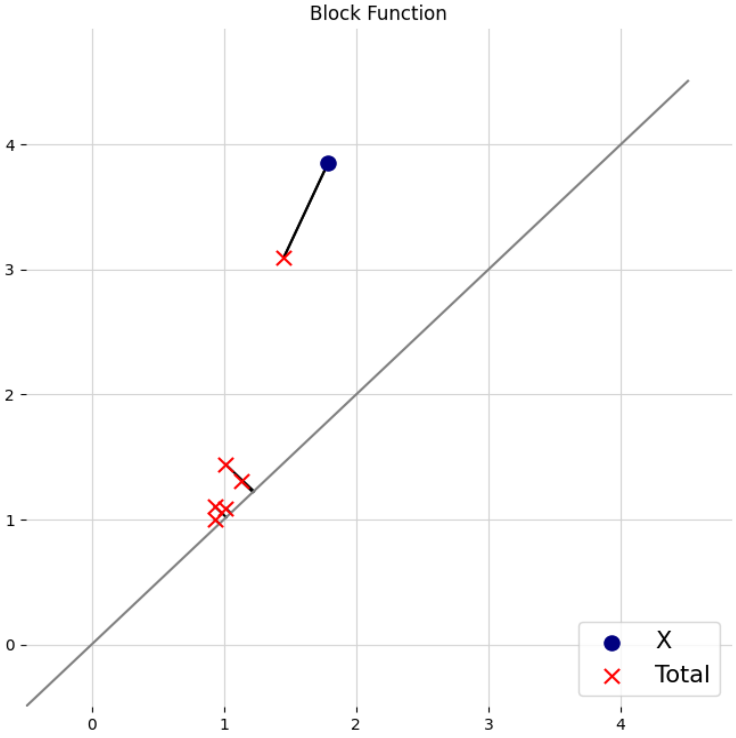

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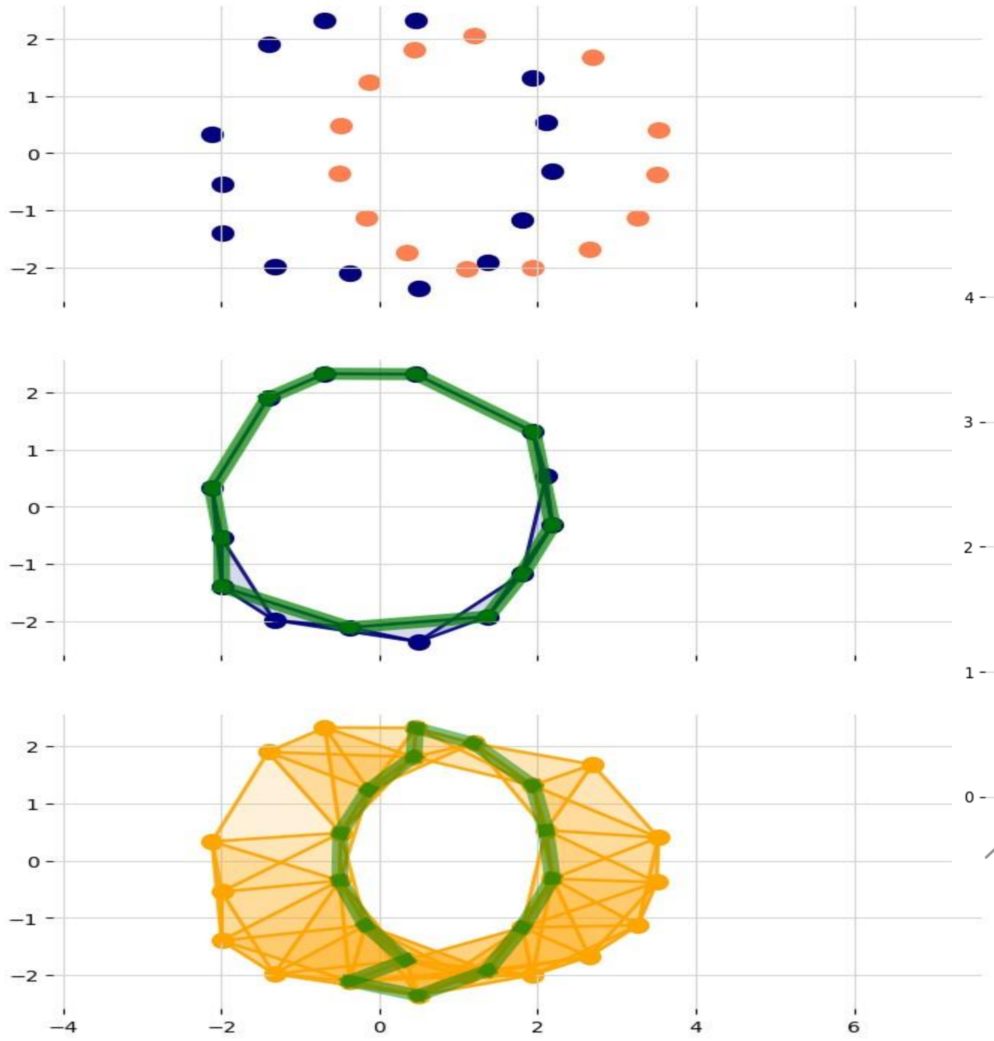

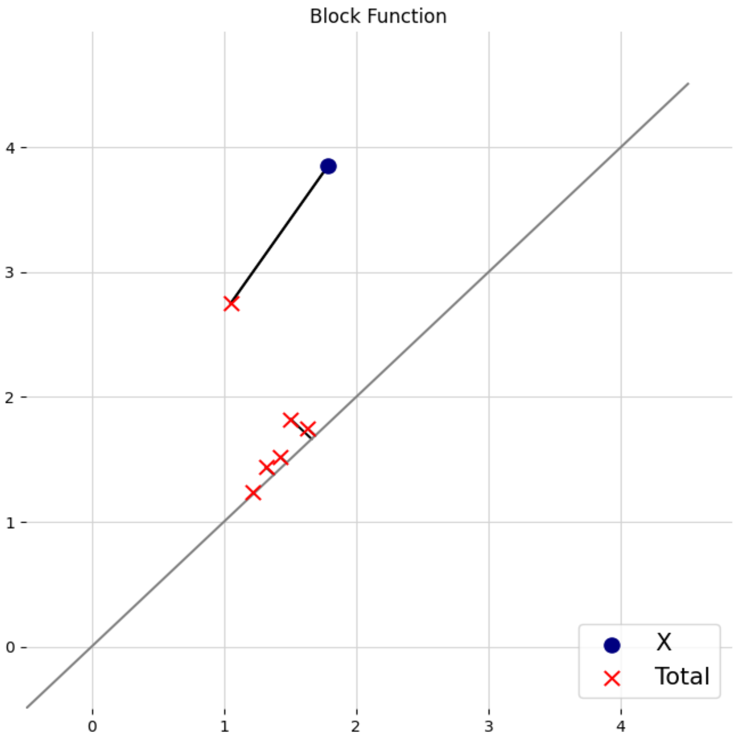

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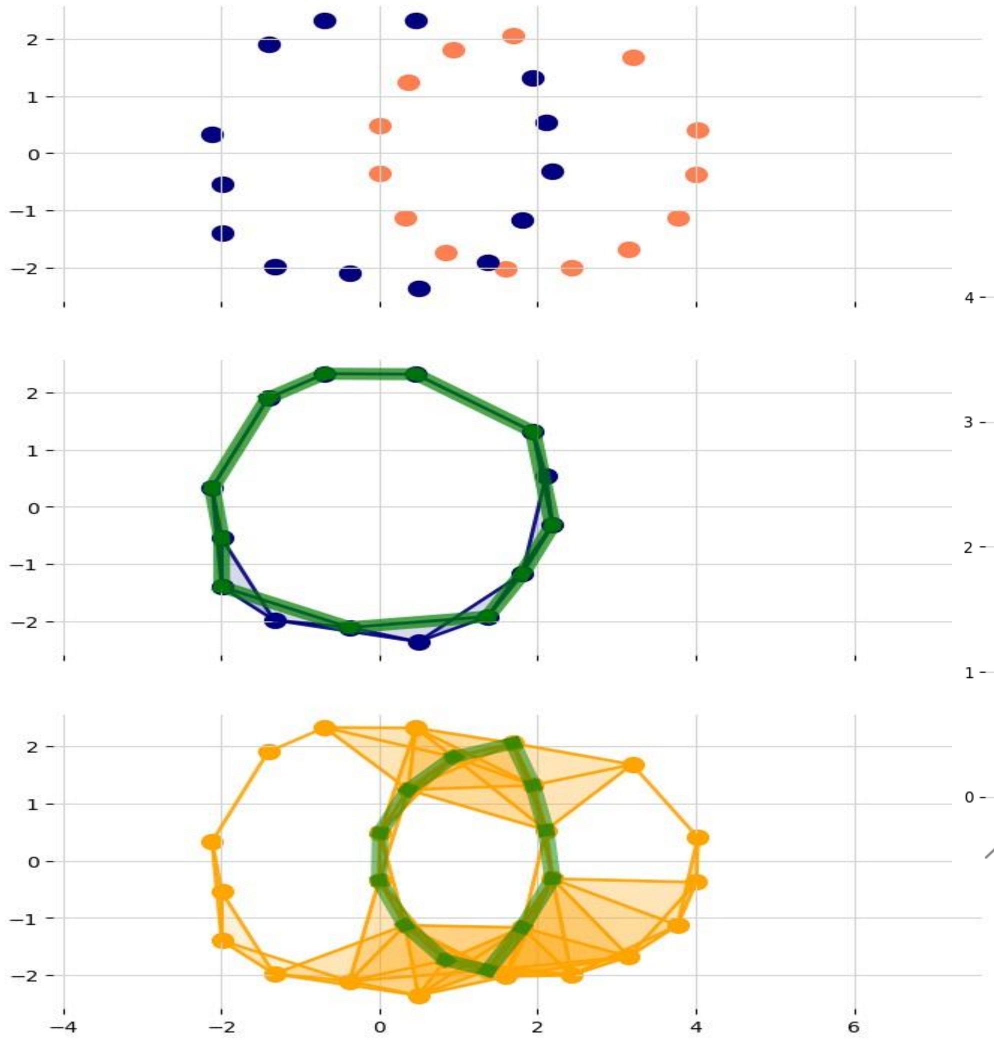

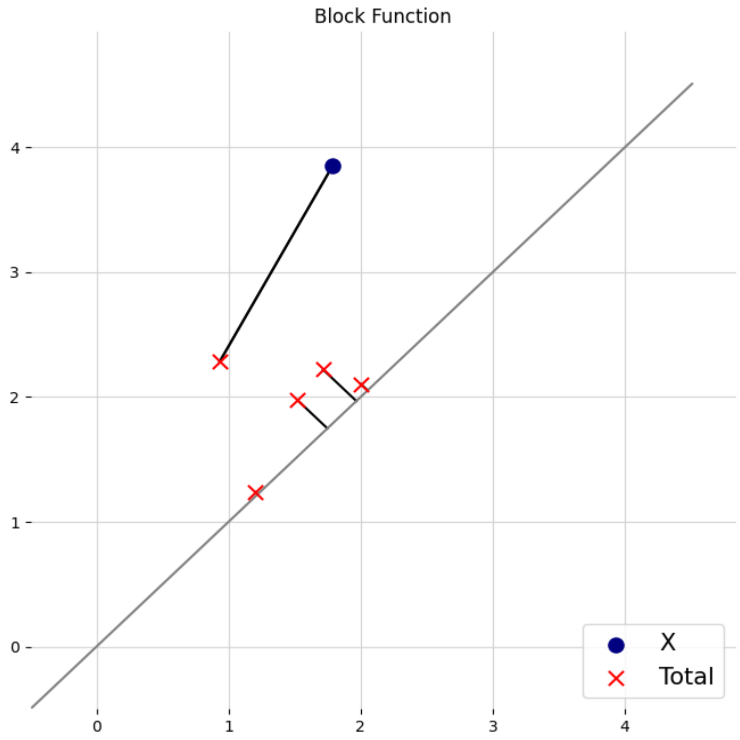

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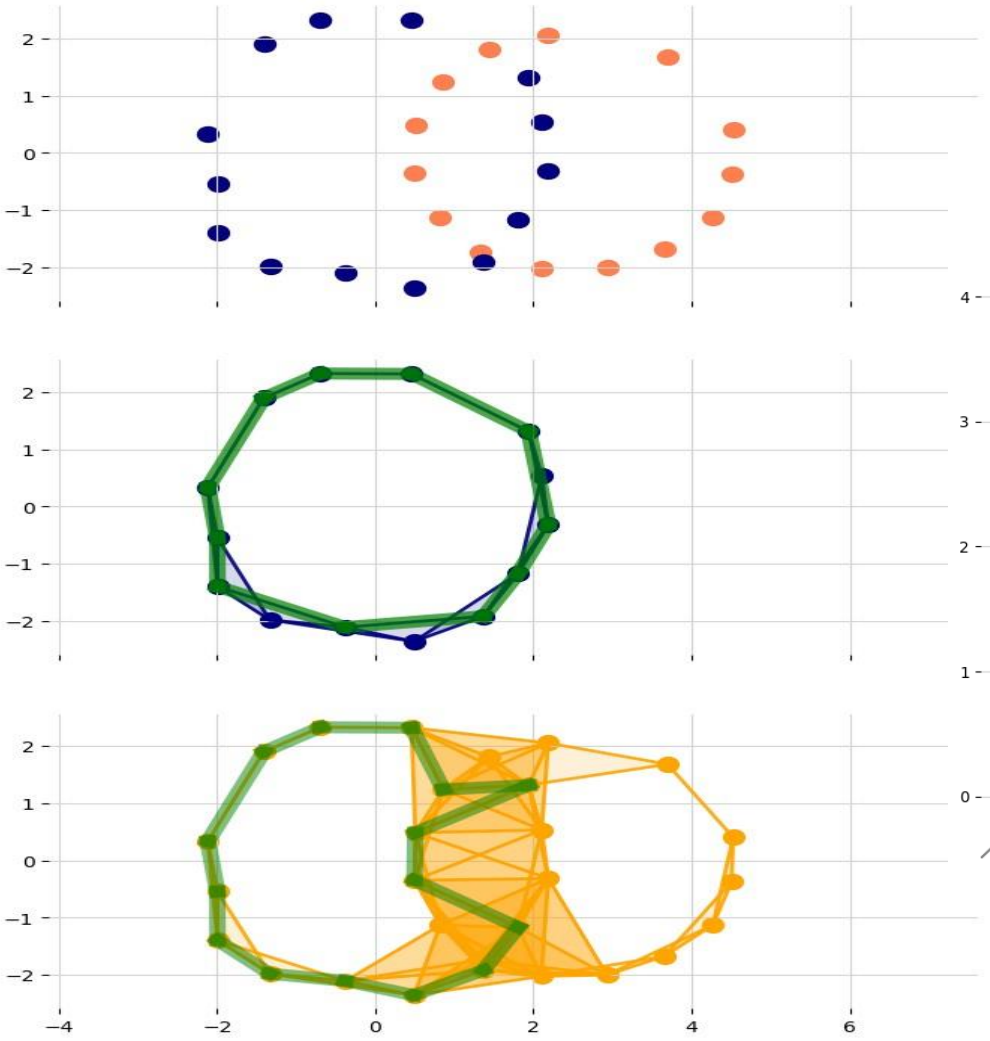

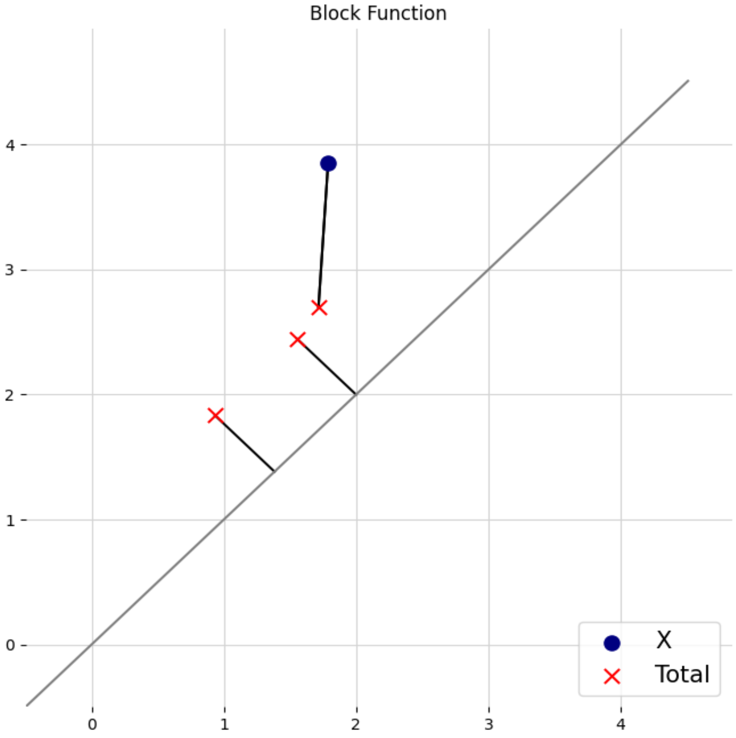

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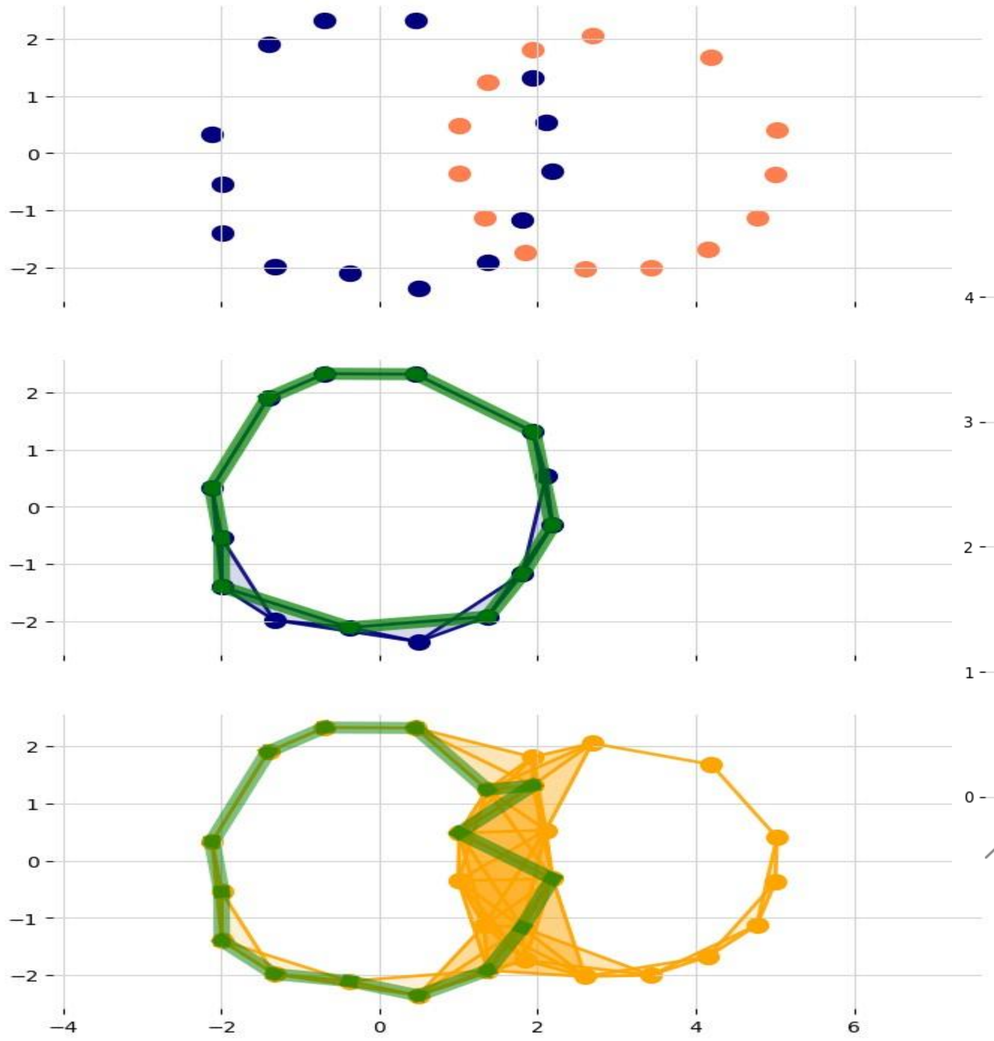
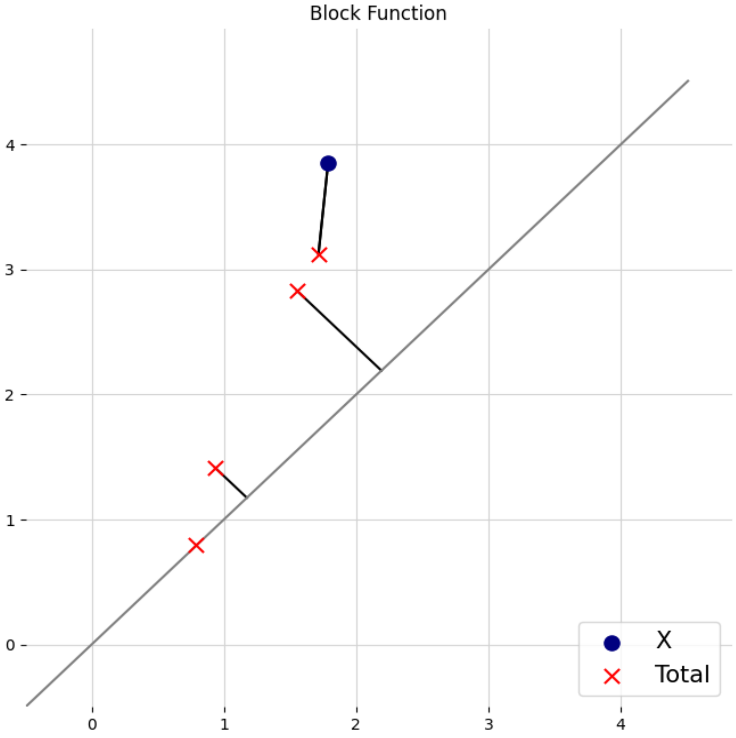
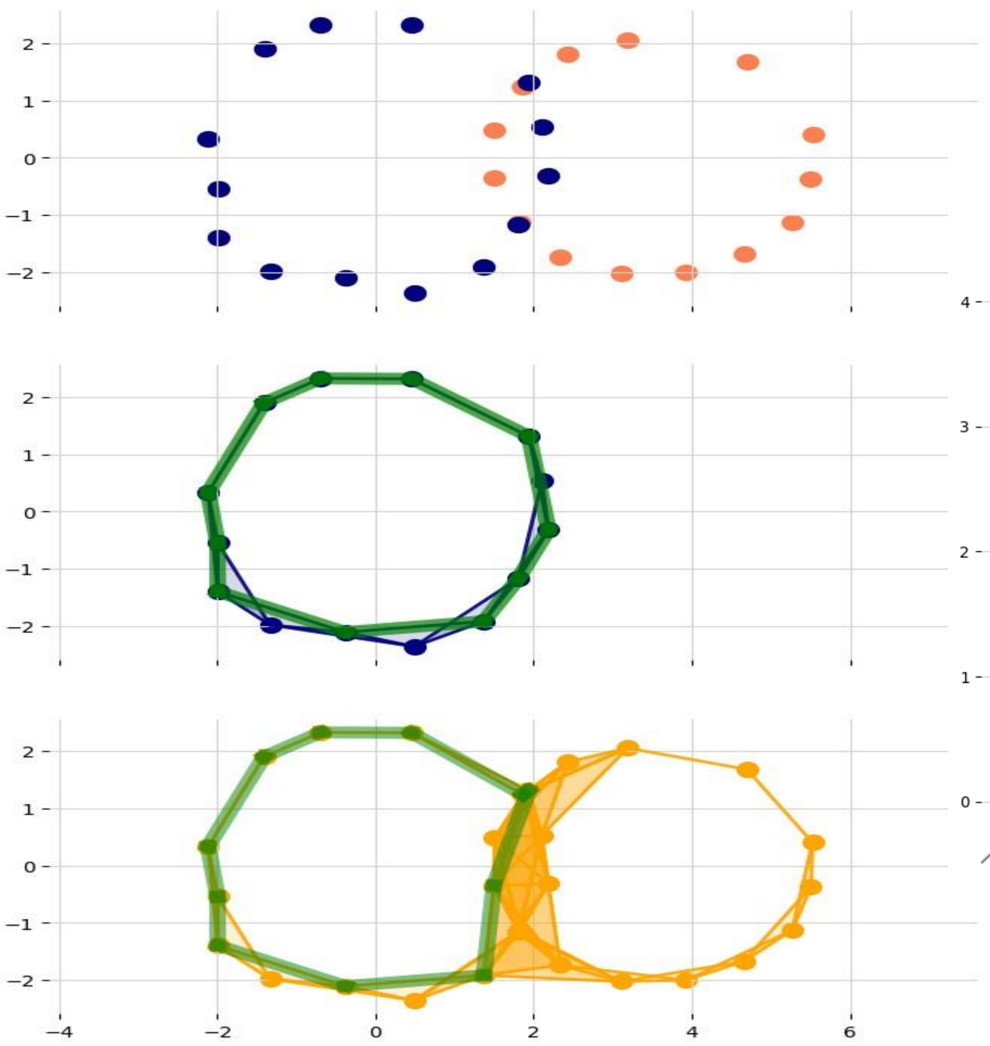
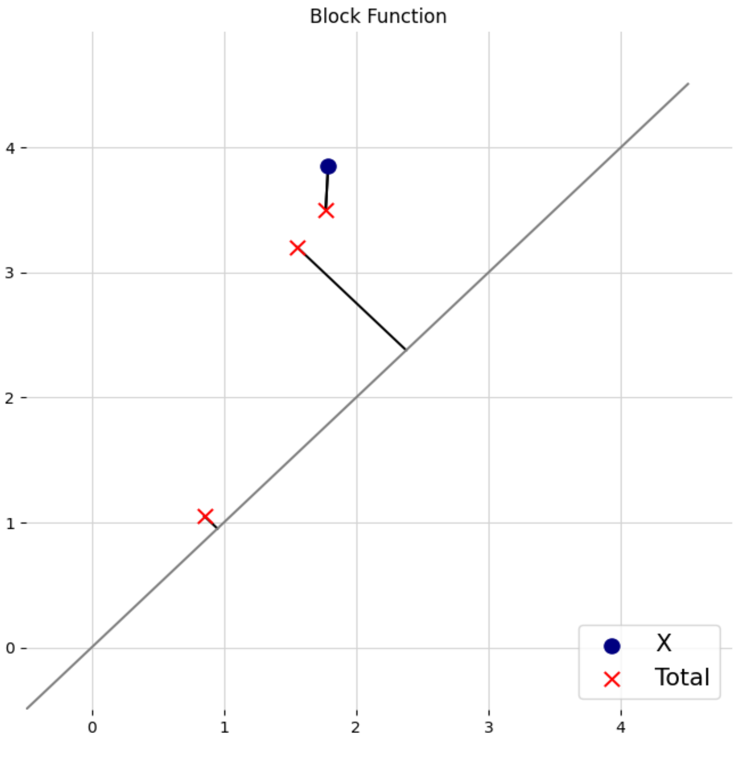




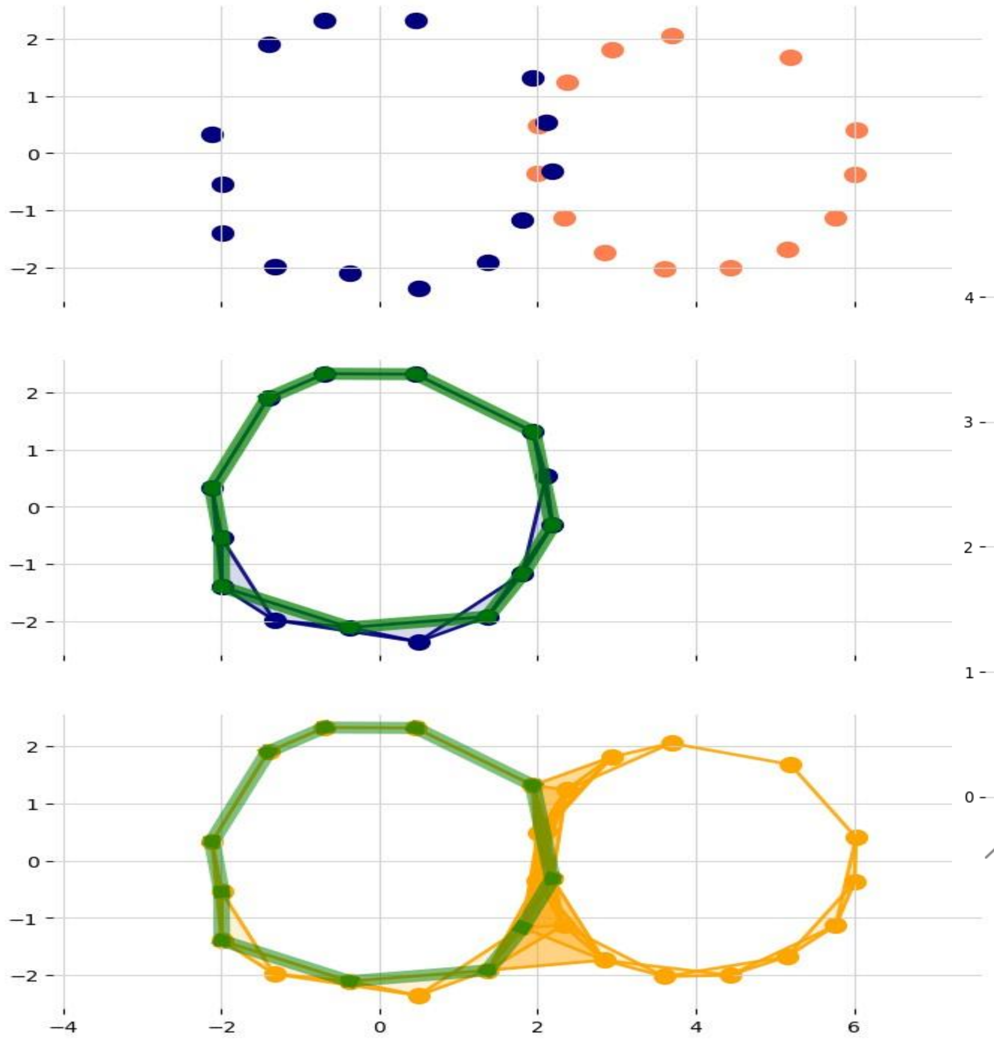
Matching circles in the plane

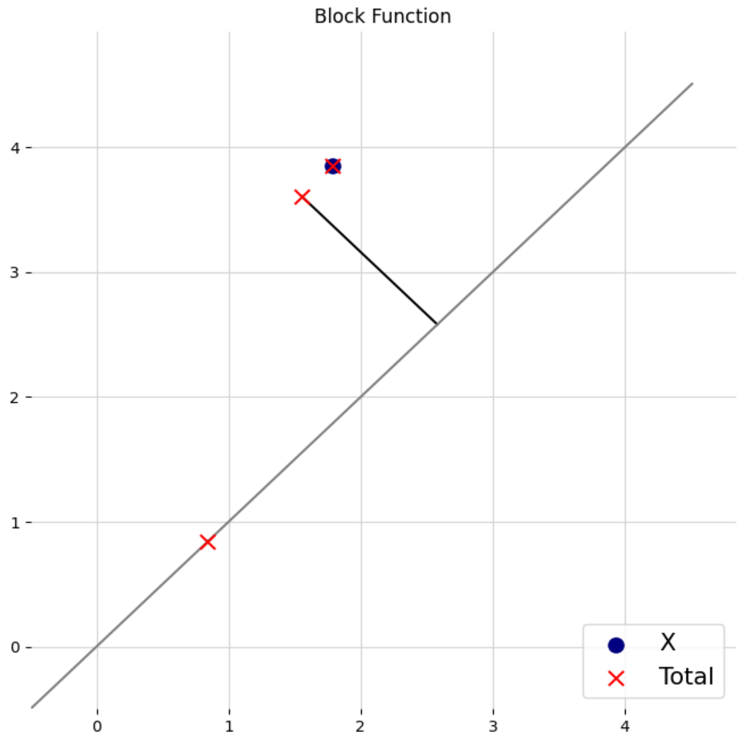
Partial matchings between barcodes





Partial matchings between barcodes





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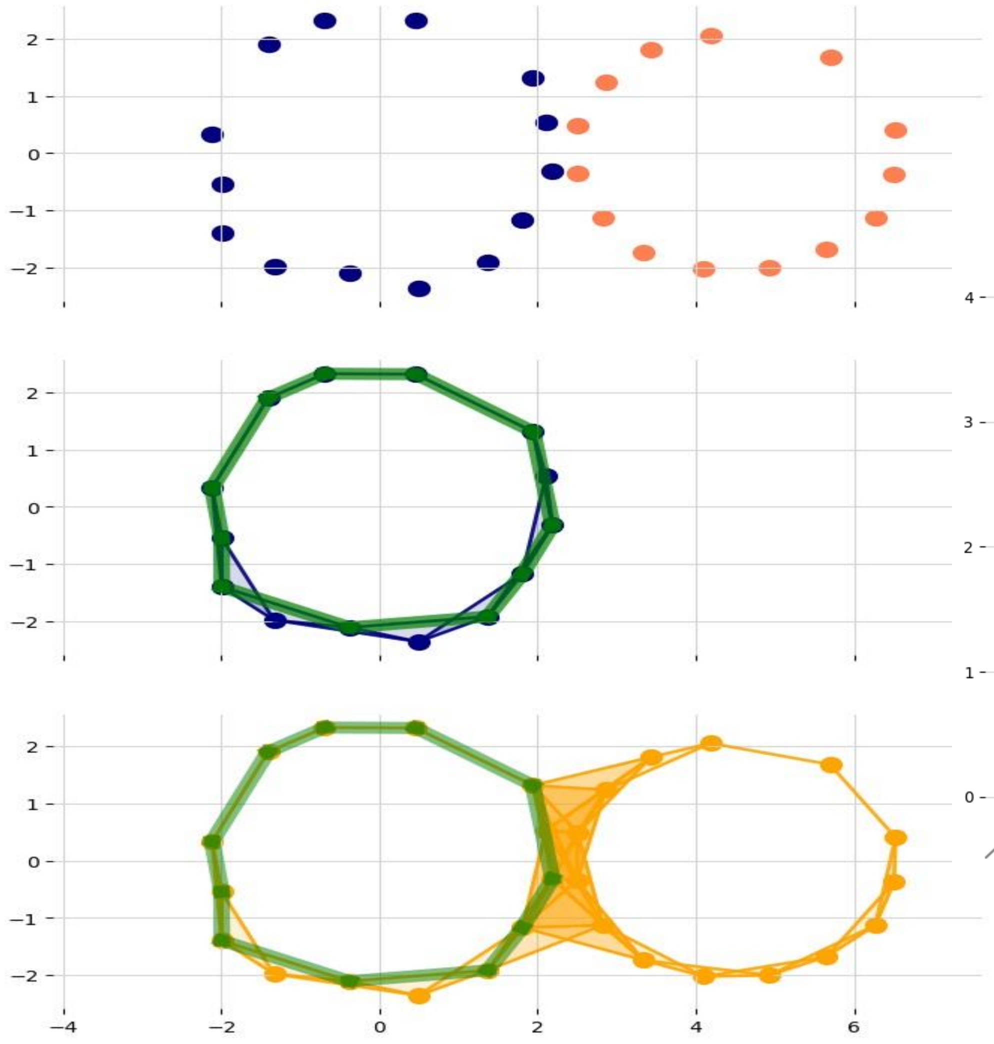

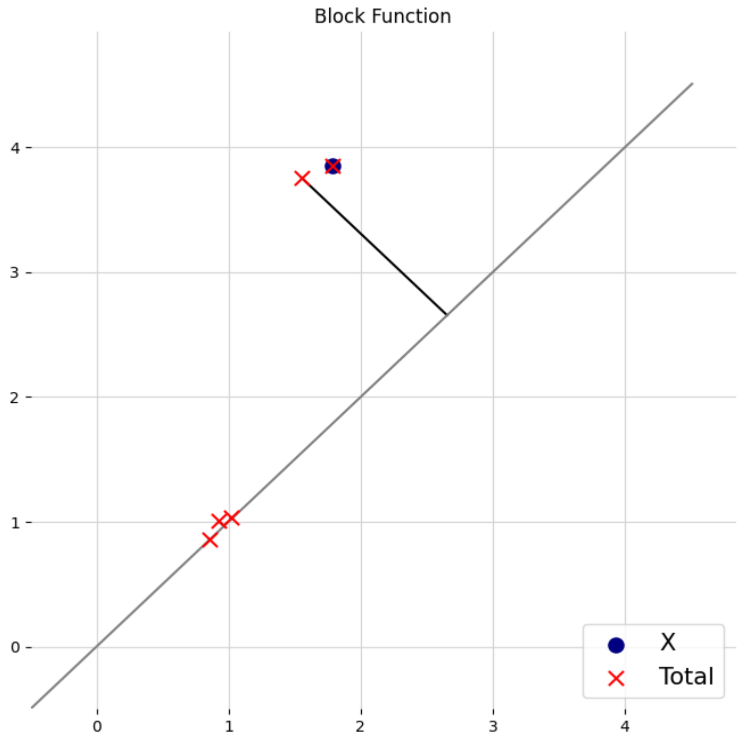

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Partial matchings between barcodes

Matrix computation:

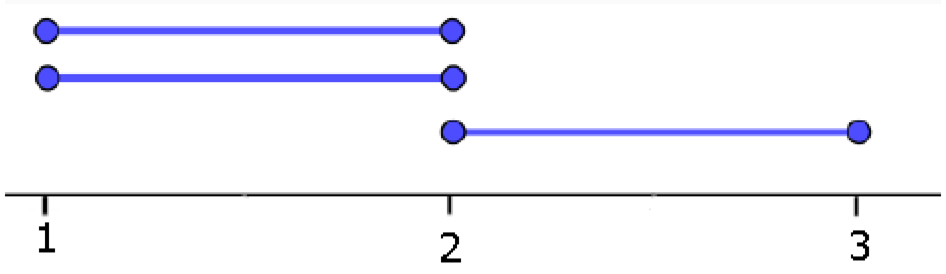
$$U \simeq k_{[1,2]} \oplus k_{[1,2]} \oplus k_{[2,3]}$$

$$\mathbf{B}(U) = \{([1,2],2),([2,3],1)\}$$

$$S_U = \{[1,2],[2,3]\}$$

Endpoint order: $[a_1, b_1] \leq [a_2, b_2]$ iff $b_1 < \mathrm{b}_2$ or $b_1 = b_2$ and $a_1 \leq a_2$

$$Rep\ B(U) = \langle [1,2],[1,2],[2,3] \rangle$$

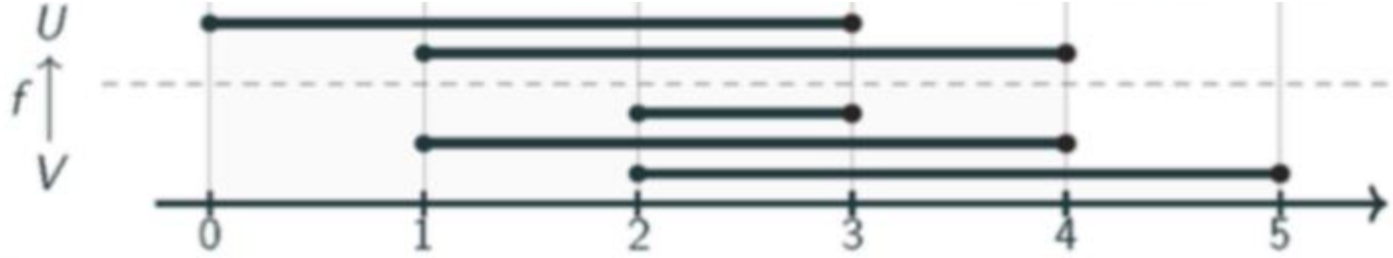


Partial matchings between barcodes

Endpoint order: $[a_1, b_1] \leq [a_2, b_2]$ iff $b_1 < \text{b}_2$ or $b_1 = b_2$ and $a_1 \leq a_2$

Matrix computation:

- Consider $V \simeq k_{[2,3]} \oplus k_{[1,4]} \oplus k_{[2,5]}$ and $U \simeq k_{[0,3]} \oplus k_{[1,4]}$.
- Order the intervals in $Rep\ B(V)$ and $Rep\ B(V)$ following the endpoint order:



- Suppose f associated to the following matrix:

$$F = \begin{array}{c|ccc} & [2,3] & [1,4] & [2,5] \\ \hline [0,3] & 1 & 1 & 0 \\ [1,4] & 0 & 1 & 1 \end{array}$$

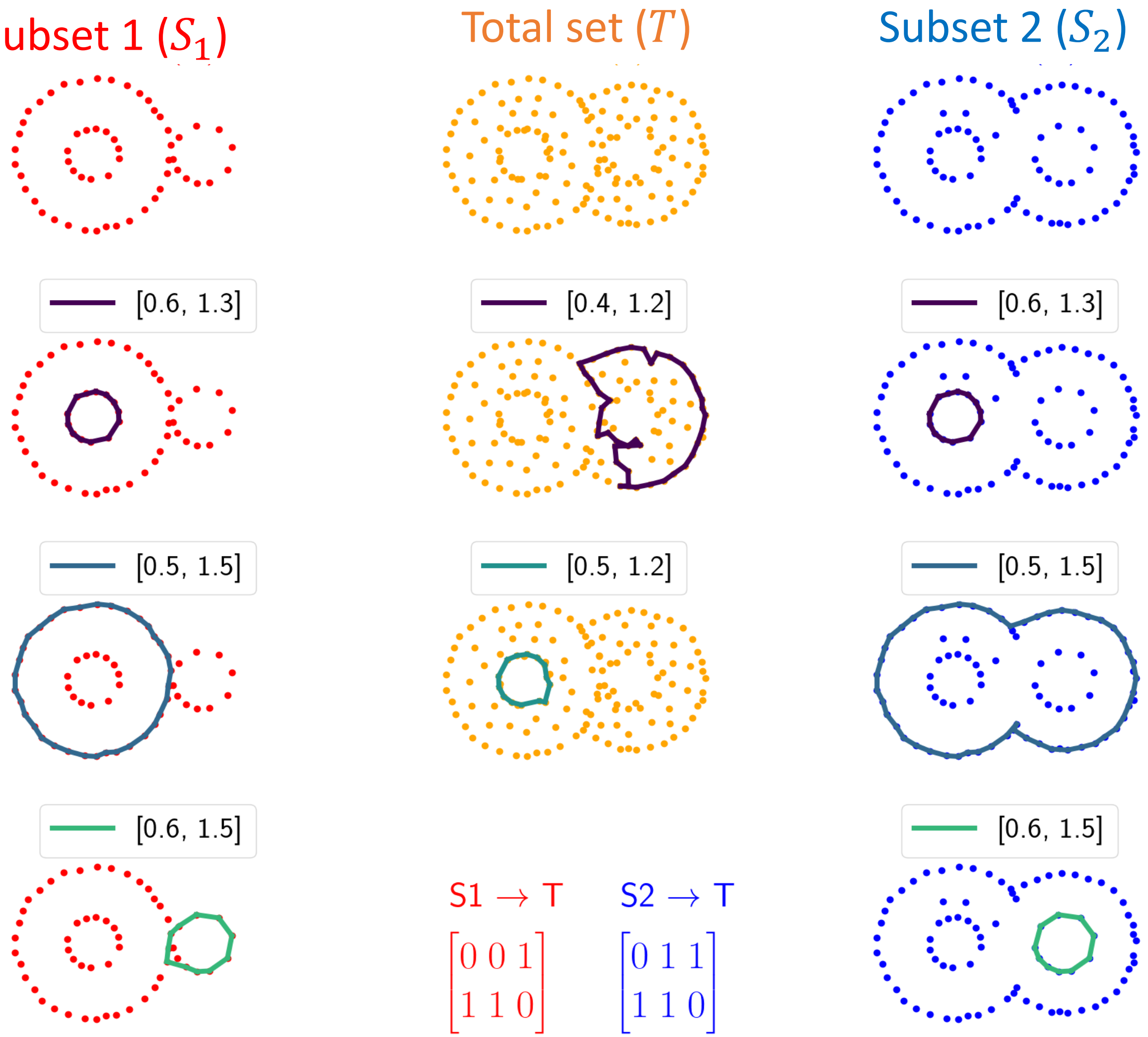
- Let $I = [a, b]$. Consider F_I, the reduced minor of F restriced to $[c, d] \in Rep\ B(U)$ with c $\leq a$ and d $\leq b$ and $[a_1, b_1] \in Rep\ B(V)$ with $b_1 < b$ or $b_1 = b$ and $a_1 \leq a$.

$$F_{[2,3]} = \begin{bmatrix} 1 \\ \ \end{bmatrix}, \quad F_{[1,4]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ and } F_{[2,5]} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- $\mathcal{M}_f$ is given by $[2,3] \mapsto [0,3]$ and $[1,4] \mapsto [1,4]$ and $[2,5] \mapsto \emptyset$.

Partial matchings between barcodes

Example:



S1 $\rightarrow$ T

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

S2 $\rightarrow$ T

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Partial matchings between barcodes

Endpoint order: $[a_1, b_1] \leq [a_2, b_2]$ iff $b_1 < b_2$ or $b_1 = b_2$ and $a_1 \leq a_2$

Example:

- Sort both $Rep\ B(S_1)$ and $Rep\ B(T)$ by the endpoint order
- We have the matrix

$$F = \left[\begin{array}{c|ccc} & [0.6, 1.3] & [0.5, 1.5] & [0.6, 1.5] \\ \hline [0.4, 1.2] & 0 & 0 & 1 \\ [0.5, 1.2] & 1 & 1 & 0 \end{array}\right]$$

- We obtain the matrices

$$F_{[0.6,1.3]} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \quad F_{[0.5,1.5]} = \begin{bmatrix} 0 \\ \mathbf{1} \end{bmatrix}, \quad F_{[0.6,1.5]} = \begin{bmatrix} 0 & 0 & \mathbf{1} \\ 1 & 0 & 0 \end{bmatrix},$$

- Assignment: $[0.6, 1.3] \mapsto [0.5, 1.2]$, $[0.5, 1.5] \mapsto [0.5, 1.2]$ and $[0.6, 1.5] \mapsto [0.4, 1.2]$.

Partial matchings between barcodes

Endpoint order: $[a_1, b_1] \leq [a_2, b_2]$ iff $b_1 < b_2$ or $b_1 = b_2$ and $a_1 \leq a_2$

Example:

- Sort both $Rep\ B(S_2)$ and $Rep\ B(T)$ by the endpoint order
- We have the matrix

$$F = \left[\begin{array}{c|ccc} & [0.6, 1.3] & [0.5, 1.5] & [0.6, 1.5] \\ \hline [0.4, 1.2] & 0 & 1 & 1 \\ [0.5, 1.2] & 1 & 1 & 0 \end{array}\right]$$

- We obtain the matrices

$$F_{[0.6,1.3]} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F_{[0.5,1.5]} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad F_{[0.6,1.5]} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

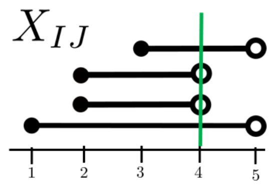
- Assignment: $[0.6, 1.3] \mapsto [0.5, 1.2]$ and $[0.5, 1.5] \mapsto [0.5, 1.2]$.

Partial matchings between barcodes

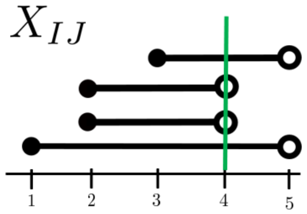
$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \sum_{J \in S_U} \mathcal{M}_f(I,J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof: Decorated points and persistence modules:



$$\dim \varinjlim_{t \in (\cdot,4]} X_{IJt} = \dim X_{IJ4} = 2$$



$$\dim \varinjlim_{t \in (\cdot,4)} X_{IJt} = 4$$

We will represent the intervals using pairs, (a, b), of elements of

$$\mathbf{E} = (\mathbb{R} \times \{-,+\}) \cup \{-\infty, +\infty\}$$

$a \backslash b$	s^-	s^+	∞
$-\infty$	$(-\infty, s)$	$(-\infty, s]$	$(-\infty, \infty)$
r^-	$[r, s)$	$[r, s]$	$[r, \infty)$
r^+	(r, s)	$(r, s]$	(r, ∞)

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \qquad \sum_{J \in S_U} \mathcal{M}_f(I,J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof:

A **section** of a vector space, $\mathcal{V}$, is a pair of vector spaces, (F^-, F^+), such that

$$F^- \hookrightarrow F^+ \hookrightarrow \mathcal{V}$$

We say that a set of sections, $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$, of $\mathcal{V}$ is **disjoint** if, for all $\lambda \neq \mu$,

$$F^+_\mu \hookrightarrow F^-_\lambda \quad \text{or} \quad F^+_\lambda \hookrightarrow F^-_\mu$$

Lemma. If $\{(F^-_\lambda, F^+_\lambda) : \lambda \in \Lambda\}$ is a set of disjoint sections of $\mathcal{V}$, we have that

$$\bigoplus_{\lambda \in \Lambda} (F^+_\lambda / F^-_\lambda) \hookrightarrow \mathcal{V}$$

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \textstyle\sum_{J \in S_U} \mathcal{M}_f(I,J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof:

$\dim\ V^+_{(s,r)\ t}$ nº of intervals (a,b) such that $t \in (a,b)$, $a \leq s$ and $b \leq r$

$\dim\ V^-_{(s,r)\ t}$ nº of intervals (a,b) such that $t \in (a,b)$, $(a < s$ and $b \leq r)$ or $(a \leq s$ and $b < r)$

$V^-_{(s,r)\ t}$ and $V^+_{(s,r)\ t}$ are persistence submodules of V and , for each t , $\left\{\left(V^-_{(s,r)\ t}, V^+_{(s,r)\ t}\right)\right\}_{s<r}$ are disjoint sections of V_t

$$\boxed{\bigoplus_{s<r} \left(V^+_{(s,r)\ t} / V^-_{(s,r)\ t}\right) \hookrightarrow V_t}$$

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof:

$$\overset{\text{fixed}}{I = (a, b) \in S_V} \qquad \overset{\text{variable}}{J = (c, d) \in S_U}$$

$$A^d_{ct} := \frac{fV^-_{It} \cap U^+_{(c,d)t} + fV^+_{It} \cap U^-_{(c,d)t}}{fV^-_{It} \cap U^+_{(c,d)t}} \quad \text{and} \quad B^d_{ct} := \frac{fV^+_{It} \cap U^+_{(c,d)t}}{fV^-_{It} \cap U^+_{(c,d)t}}$$

$$\frac{B^d_{ct}}{A^d_{ct}} \simeq X_{IJt}$$

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof:

$$\tilde{A}^d_c := \varinjlim_{t \in I \cap J} A^d_{ct} \qquad \tilde{B}^d_c := \varinjlim_{t \in I \cap J} B^d_{ct}$$

- For fixed d and variable c, they are persistence modules indexed by $\mathbf{E}$
- They satisfies that

$$\frac{\tilde{B}^d_c}{\tilde{A}^d_c} \simeq \varinjlim_{t \in I \cap J} X_{IJt}$$

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad\qquad \textstyle\sum_{J \in S_U} \mathcal{M}_f(I, J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I, J) = \dim \varinjlim_{t \in I \cap J} X_{IJt}}$$

Proof:

For each d,

$\left\{(\tilde{A}_c^d, \tilde{B}_c^d) : c \in \mathbf{E}\right\}$ is a disjoint set of sections of $\varinjlim_{c<d} \tilde{B}_c^d$

$$\left(\bigoplus_{c<d} \underbrace{\varinjlim_{t \in I \cap J} X_{I(c,d)t}}_{\frac{\tilde{B}_c^d}{\tilde{A}_c^d}}\right) \hookrightarrow \varinjlim_{c<d} \tilde{B}_c^d \hookrightarrow \varinjlim_{t \in (-\infty, d)} \left(f V_{It}^+ / f V_{It}^-\right)$$

Partial matchings between barcodes

$$\mathcal{M}_f : S_V \times S_U \to \mathbb{Z}_{\geq 0} \qquad \textstyle\sum_{J\in S_U} \mathcal{M}_f(I,J) \leq m_I$$

$$\boxed{\mathcal{M}_f(I,J) = \dim \varinjlim_{t\in I\cap J} X_{IJt}}$$

Proof:

$$\left(\bigoplus_{c<d} \varinjlim_{t\in I\cap J} X_{I(c,d)t}\right) \hookrightarrow \varinjlim_{c<d} \tilde{B}^d_c$$

$$\varinjlim_{c<d} \tilde{B}^d_c \hookrightarrow \varinjlim_{t\in(-\infty,d)} \left(fV^+_{It}/fV^-_{It}\right)$$

$$\textstyle\sum_{J\in S_U} \mathcal{M}_f(I,J)$$

$$\|$$

$$\sum_{d\leq b} \dim\left(\bigoplus_{c<d} \varinjlim_{t\in(-\infty,d)} X_{I(c,d)t}\right)$$

$$\leq \sum_{d\leq b} \dim \varinjlim_{c<d} \tilde{B}^d_c$$

$\leq$ # intervals in fV^+_{It}/fV^-_{It} $\square$

$\leq$ # intervals in $V^+_{It}/V^-_{It} = m_I$

Links

https://doi.org/10.1016/j.comgeo.2023.101985

https://arxiv.org/abs/2306.02411

Decomposition of pointwise finite-dimensional persistence modules. W. Crawley-Boevey.

Topology Tools
for
Explainable and Green Artificial Intelligence

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- Context: Green and Explainable artificial intelligence (REXASI-PRO)
- Computational topology tools: Persistent homology, barcodes, distance bottleneck, simplicial maps, Persistence modules, morphisms between persistence modules
- Partial matchings between barcodes
- Simplicial maps neural networks